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QUASI-EXACTLY-SOLVABLE DIFFERENTIAL EQUATIONS

ALEXANDER TURBINER

Abstract. A general classification of linear differential and finite-difference operators possessing a finite-dimensional invariant subspace with a polynomial basis is given. The main result is that any operator with the above property must have a representation as a polynomial element of the universal enveloping algebra of the algebra of differential (difference) operators in finite-dimensional representation. In one-dimensional case a classification is given by algebras $sl_2(\mathbf{R})$ (for differential operators in $\mathbf{R}$) and $sl_2(\mathbf{R})_q$ (for finite-difference operators in $\mathbf{R}$), $osp(2,2)$ (operators in one real and one Grassmann variable, or equivalently, 2×2 matrix operators in $\mathbf{R}$) and $gl_2(\mathbf{R})_K$ (for the operators containing the differential operators and the parity operator). A classification of linear operators possessing infinitely many finite-dimensional invariant subspaces with a basis in polynomials is presented.

Contents

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This Chapter is devoted to a description of a new connection between Lie algebras and linear differential equations.

The main idea is surprisingly easy. Let us consider a certain set of differential operators of the first order

$$J^\alpha(x) \ = \ a^{\alpha,\mu}(x)\partial_\mu \ + \ b^\alpha(x) \quad , \quad \partial_\mu \equiv \frac{d}{dx^\mu} \tag{0}$$

$\alpha = 1, 2, \ldots, k$, $x \in R^n$, $\mu = 1, 2, \ldots, n$ and $a^{\alpha,\mu}(x)$, $b^\alpha(x)$ are certain functions on R^n. Then assume that the operators form a basis of some Lie algebra g of the dimension $k = \dim g$. Now take a polynomial h in generators $J^\alpha(x)$ and ask a question:

> Does the differential operator $h(J^\alpha(x))$ have some specific properties, which distinguish this operator from a general linear differential operator?

Generically, an answer is *negative.* However, if the algebra g is taken in a finite-dimensional representation, the answer becomes *positive* :

> The differential operator $h(J^\alpha(x))$ does possess a finite-dimensional invariant subspace coinciding a representation space of the finite-dimensional representation of the algebra g of differential operators of the first order. If a basis of this finite-dimensional representation space can be constructed explicitly, the operator h can be presented in the explicit block-diagonal form.

Such differential operators having a finite-dimensional invariant subspace with an explicit basis in functions can be named *quasi-exactly-solvable.*

Up to our knowledge and understanding the first *explicit* examples of quasi-exactly-solvable problems were found by Razavy [1980, 1981] and by Singh-Rampal-Biswas-Datta [1980]. In an explicit form a general idea of quasi-exactly-solvability had been formulated at the first time in Turbiner [1988a, 1988b] and it had led to a complete catalogue of one-dimensional, quasi-exactly-solvable Schroedinger operators in connection with spaces of polynomials Turbiner [1988c]. The term "quasi-exactly-solvability" has been suggested in Turbiner-Ushveridze [1987]. A connection of quasi-exactly-solvability and finite-dimensional representations of sl_2 was mentioned at the first time by Zaslavskii-Ulyanov [1984]. Later, the idea of quasi-exactly-solvability was generalized to multidimensional differential operators, matrix differential operators (Shifman and Turbiner [1989]), finite-difference operators (Ogievetsky and Turbiner [1991]) and, recently, to "mixed" operators containing differential operators and permutation operators (Turbiner [1994a]).

In Morozov et al [1990] it was described a connection of quasi-exactly-solvability with conformal quantum field theories (see also Halpern-Kiritsis [1989]), while recent development can be found in Tseytlin [1994] and also in a brief review done by Shifman [1994]. Relationship with solid-state physics is given in (Ulyanov and Zaslavskii [1992]), while recently very exciting results in this direction were found by Wiegmann-Zabrodin [1994a, 1994b]. A general survey of a phenomenon of the quasi-exactly-solvability was done in Turbiner [1994b].

Given Chapter will be devoted mainly to a description of quasi-exactly-solvable operators acting on functions in one real (complex) variable.

1. Generalities

Let us take n linearly-independent functions $f_1(x), f_2(x) \dots f_n(x)$ and form a linear space

$$\mathcal{F}_n(x) \ = \ \langle f_1(x), f_2(x), \dots, f_n(x)\rangle, \tag{1.1}$$

where n is a non-negative integer and $x \in \mathbf{R}$.

Definition 1.1. Two spaces $\mathcal{F}_n^{(1)}(x)$ and $\mathcal{F}_n^{(2)}(x)$ are named *equivalent* spaces, if one space can be obtained through another one via a change of the variable and/or the multiplication on a some function (making a gauge transformation),

$$\mathcal{F}_n^{(2)}(x) = g(x)\mathcal{F}_n^{(1)}(y(x)) \tag{1.2}$$

Choosing a certain space (1.1) and considering all possible changes of the variable, $y(x)$ and gauge functions, $g(x)$, one can describe a whole class of equivalent spaces. It allows to introduce a certain standartization: $f_1(x) = 1, f_2(x) = x$, since in each class of spaces, one can find a representative satisfying the standartization. Thus, hereafter the only spaces of the following form

$$\mathcal{F}_n(x) \ = \ \langle 1, x, f_1(x), f_2(x), \dots, f_n(x)\rangle, \tag{1.3}$$

are taking into account.

It is easy to see that once an operator $h(y, d_y)$ acts on a space $\mathcal{F}_n^{(1)}(y)$, one can construct an operator acting on an equivalent space $\mathcal{F}_n^{(2)}(y(x))$

$$\bar{h}(y, d_y) = g(x)h(x, d_x)g^{-1}(x)|_{x=x(y)} \quad , \quad d_x \equiv \frac{d}{dx} \ . \tag{1.4}$$

Later it will be considered one of the most important particular case of the space (1.3): the space of polynomials of finite order

$$\mathcal{P}_{n+1}(x) \ = \ \langle 1, x, x^2, \dots, x^n\rangle, \tag{1.5}$$

where n is a non-negative integer and $x \in \mathbf{R}$. It is worth noting the important property of an invariance of the space (1.5):

$$x^n\mathcal{P}_{n+1}(\frac{1}{x}) \ \equiv \ \mathcal{P}_{n+1}(x), \tag{1.6}$$

It stems from an evident feature of polynomials: let $p_n(x) \in \mathcal{P}_{n+1}(x)$ is any polynomial, then $x^n p_n(1/x)$ remains a polynomial with inverse order of the coefficients then in $p_n(x)$.

2. Ordinary differential equations

2.1. General consideration.

Definition 2.1. Let us name a linear differential operator of the kth order, $T_k(x, d_x)$ *quasi-exactly-solvable*, if it preserves the space $\mathcal{P}_{n+1}$. Correspondingly, the operator $E_k(x, d_x)$ is named *exactly-solvable*, if it preserves the infinite flag $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \dots \subset \mathcal{P}_n \subset \dots$ of spaces of all polynomials: $E_k(x, d_x) : \mathcal{P}_j \mapsto \mathcal{P}_j \ , \ j = 0, 1, \dots .$

Lemma 2.1. *(Turbiner* [1994b]*) (i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator T_k can be represented by a k-th degree polynomial of the operators*

$$J_n^+ = x^2 d_x - nx,$$

$$J_n^0 = x d_x - \frac{n}{2}\ , \tag{2.1}$$

$$J_n^- = d_x\ ,$$

(the operators (2.1) obey the $sl_2(\mathbf{R})$ commutation relations: $[J^{\pm}, J^0] = \pm J^{\pm}$, $[J^+, J^-] = -2J^0$ [1]). If $n \leq (k-1)$, the part of the quasi-exactly-solvable operator T_k containing derivatives up to order n can be represented by an nth degree polynomial in the generators (2.1).

(ii) Conversely, any polynomial in (2.1) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k$.

Definition 2.2. Let us name a *universal enveloping algebra* U_g of a Lie algebra g the algebra of all ordered polynomials in generators $J^{\pm,0}$. The notion *ordering* means that in any monomial in generators $J^{\pm,0}$ all J^+ are placed to the left and all J^- to the right.

Comment 2.1 . Notion the universal enveloping algebra allows to make a statement that T_k at $k < n+1$ is simply an element of the universal enveloping algebra $U_{sl_2(\mathbf{R})}$ of the algebra $sl_2(\mathbf{R})$ taken in representation (2). If $k \geq n+1$, then T_k is represented as an element of $U_{sl_2(\mathbf{R})}$ plus $B\frac{d^{n+1}}{dx^{n+1}}$, where B is any linear differential operator of order not higher than $(k-n-1)$. In otherwords, the algebra of differential operators acting on the space (1.4) coincides to the universal enveloping algebra $U_{sl_2(\mathbf{R})}$ of the algebra $sl_2(\mathbf{R})$ taken in representation (2.1) plus operators annihilating (1.4).

Comment 2.2 . The algebra (2.1) has the following invariance property

$$x^{-n}\ J_n^{\pm,0}(x, d_x)\ x^n\ |_{x=\frac{1}{z}}\ \Rightarrow\ J_n^{\pm,0}(z, d_z)$$

as a consequence of the invariance (1.6) of the space $\mathcal{P}_{n+1}$. In particular, if $z = 1/x$, then

$$\begin{aligned} J_n^+(x, d_x) &\Rightarrow -J_n^-(z, d_z) \\ J_n^0(x, d_x) &\Rightarrow -J_n^0(z, d_z) \\ J_n^-(x, d_x) &\Rightarrow -J_n^+(z, d_z) \end{aligned} \tag{2.2}$$

Let us introduce the *grading* of generators (2.1) as follows. It is easy to check that any $sl_2(\mathbf{R})$-generator maps a monomial into monomial, $J_n^{\alpha} x^p \mapsto x^{p+d_\alpha}$.

Definition 2.3. The number d_α is named a *grading* of the generator J_n^α: $deg(J_n^\alpha) = d_\alpha$.

Following this definition

$$deg(J_n^+) = +1\ ,\ deg(J_n^0) = 0\ ,\ deg(J_n^-) = -1, \tag{2.3}$$

and

$$deg[(J_n^+)^{n_+}(J_n^0)^{n_0}(J_n^-)^{n_-}]\ =\ n_+ - n_-. \tag{2.4}$$

Notion of the grading allows us to classify the operators T_k in the Lie-algebraic sense.

[1] The representation (2.1) is one of the 'projectivized' representations (see Turbiner [1988a, 1988b]). This realization of $sl_2(\mathbf{R})$ has been derived at the first time by Sophus Lie.

Lemma 2.2. *A quasi-exactly-solvable operator $T_k \subset U_{sl_2(\mathbf{R})}$ has no terms of positive grading, if and only if it is an exactly-solvable operator.*

Comment 2.3 . Any exactly-solvable operator having term of negative grading possesses terms of positive grading after transformation (2.2). A quasi-exactly-solvable operator always possesses terms of positive grading as in x-space representation, as in z-space representation.

Theorem 2.1. *(Turbiner* [1994b]*) Let n be a non-negative integer. Take the eigenvalue problem for a linear differential operator of the kth order in one variable*

$$T_k(x, d_x)\varphi(x) \ = \ \varepsilon\varphi(x) \ , \tag{2.5}$$

where T_k is symmetric. The problem (2.5) has $(n+1)$ linearly independent eigenfunctions in the form of a polynomial in variable x of order not higher than n, if and only if T_k is quasi-exactly-solvable. The problem (2.5) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable.

Comment 2.4 . The " if " part of the first statement is obvious. The " only if " part is a direct corollary of Lemma 2.1 .

This theorem gives a general classification of differential equations

$$\sum_{j=0}^{k} a_j(x) d_x^j \varphi(x) \ = \ \varepsilon\varphi(x) \tag{2.6}$$

having at least one polynomial solution in x. The coefficient functions $a_j(x)$ must have the form

$$a_j(x) \ = \ \sum_{i=0}^{k+j} a_{j,i} x^i \tag{2.7}$$

The explicit expressions (2.7) for coefficient function in (2.6) are obtained by the substitution (2.1) into a general, kth degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})}$. Thus the coefficients $a_{j,i}$ can be expressed through the coefficients of the kth degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})}$. The number of free parameters of the polynomial solutions is defined by the number of parameters characterizing a general kth degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})}$. A rather straightforward calculation leads to the following formula

$$par(T_k) = (k+1)^2 \tag{2.8}$$

where we denote the number of free parameters of operator T_k by the symbol $par(T_k)$. For the case of an infinite sequence of polynomial solutions expression (2.7) simplifies to

$$a_j(x) \ = \ \sum_{i=0}^{j} a_{j,i} x^i \tag{2.9}$$

in agreement with the results by Krall [1938] (see also Littlejohn [1988]). In this case the number of free parameters is equal to

$$par(E_k) = \frac{(k+1)(k+2)}{2} \tag{2.10}$$

On can show that the operators T_k with the coefficients (2.9) preserve a finite flag $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_k$ of spaces of polynomials. One can easily verify that the

preservation of such a finite flag of spaces of polynomial leads to the preservation of an infinite flag of such spaces.

A class of spaces equivalent to the space of polynomials (1.5) is presented by

$$\langle \alpha(z), \alpha(z)\beta(z), \ldots, \alpha(z)\beta(z)^n \rangle \ , \tag{2.11}$$

where $\alpha(z), \beta(z)$ are any functions. Linear differential operators acting in the (2.11) are easily obtained from the quasi-exactly-solvable operators (2.5)–(2.7) (see Lemma 2.1) making the change of variable $x = \beta(z)$ and the "gauge" transformation $\tilde{T} = \alpha(z) T \alpha(z)^{-1}$ and have the form

$$\bar{T}_k \ = \ \alpha(z) \sum_{j=0}^{k} (\sum_{i=0}^{k+j} a_{j,i} \beta(z)^i)(d_x^j) \mid_{x=\beta(z)} \alpha(z)^{-1} \tag{2.12}$$

where the coefficients $a_{j,i}$ are the same as in (2.7).

So the expression (2.12) gives a general form of the linear differential operator of the kth order acting a space equivalent to (1.5). Since any one- or two-dimensional invariant sub-space can be presented in the form (2.11) and can be reduced to (1.5), the general statement takes place:

Theorem 2.2. *There are no linear operators possessing one- or two-dimensional invariant sub-space with an explicit basis other than given by Lemma 2.1.*

Therefore an eigenvalue problem (2.5), for which one eigenfunction can be found in an explicit form, is related to the operator

$$\mathbf{T^{(1)}} \ = \ B(x, d_x) d_x \ + \ q_0 \tag{2.13}$$

and its all modifications (1.4), occurred after a change of the variable and a "gauge" transformation. A general differential operator possessing two eigenfunctions in an explicit form is presented as

$$\mathbf{T^{(2)}} \ = \ B(x, d_x) d_x^2 \ + \ q_2(x) d_x \ + \ q_1(x), \tag{2.14}$$

plus its all modifications owing to a change of the variable and a "gauge" transformation. Here $B(x, d_x)$ is any linear differential operator, $q_0 \in \mathbf{R}$ and $q_{1,2}(x)$ are the first- and second-order polynomials, respectively, with coefficients such that $q_2(x)d_x + q_1(x)$ can be expressed as a linear combination of the generators $J_1^{\pm,0}$ (see (2.1)).

2.2. **Second-order differential equations.** The second-order differential equations play exceptionally important role in applications. Therefore, let us consider in details the second-order differential equation (2.5) possessing polynomial solutions. From Theorem 2.1 it follows that the corresponding differential operator must be quasi-exactly-solvable and can be represented as

$$T_2 = c_{++} J_n^+ J_n^+ + c_{+0} J_n^+ J_n^0 + c_{+-} J_n^+ J_n^- + c_{0-} J_n^0 J_n^- + c_{--} J_n^- J_n^- +$$

$$c_+ J_n^+ + c_0 J_n^0 + c_- J_n^- + c, \tag{2.15.1}$$

where $c_{\alpha\beta}, c_\alpha, c \in \mathbf{R}$. The number of free parameters is $par(T_2) = 9$. Under the condition $c_{++} = c_{+0} = c_+ = 0$, the operator T_2 becomes exactly-solvable (see Lemma 2.2)

$$E_2 = c_{+-} J_n^+ J_n^- + c_{0-} J_n^0 J_n^- + c_{--} J_n^- J_n^- + c_0 J_n^0 + c_- J_n^- + c, \tag{2.15.2}$$

and the number of free parameters is reduced to $par(E_2) = 6$.

Lemma 2.3. *If the operator (2.15.1) is such that*

$$c_{++} = 0 \quad and \quad c_{+} = (\frac{n}{2} - m)c_{+0} \ , \ at\ some\ m = 0, 1, 2, \dots \tag{2.16}$$

then the operator T_2 preserves both $\mathcal{P}_{n+1}$ and $\mathcal{P}_{m+1}$. In this case the number of free parameters is $par(T_2) = 7$.

In fact, Lemma 2.3 claims that $T_2(J^\alpha_n, c_{\alpha\beta}, c_\alpha)$ can be rewritten as $T_2(J^\alpha_m, c'_{\alpha\beta}, c'_\alpha)$. As a consequence of Lemma 2.3 and Theorem 2.1, in general, among polynomial solutions of (2.6) there are polynomials of order n and order m.

Remark. From the Lie-algebraic point of view Lemma 2.3 means the existence of representations of second-degree polynomials in the generators (2.1) possessing two invariant sub-spaces. In general, if n in (2.1) is a non-negative integer, then among representations of kth degree polynomials in the generators (2.1), lying in the universal enveloping algebra, there exist representations possessing $1, 2, ..., k$ invariant sub-spaces. Even starting from an infinite-dimensional representation of the original algebra (n in (2.1) is *not* a non-negative integer), one can construct the elements of the universal enveloping algebra having finite-dimensional representation (e.g., the parameter n in (2.16) is non-integer, however T_2 has the invariant sub-space of dimension $(m + 1)$). Also this property implies the existence of representations of the polynomial elements of the universal enveloping algebra $U_{sl_2(\mathbf{R})}$, which can be obtained starting from different representations of the original algebra (2.1).

Substituting (2.1) into (2.15.1) and then into (2.6), we obtain

$$-P_4(x)d_x^2\varphi(x) \ + \ P_3(x)d_x\varphi(x) \ + \ P_2(x)\varphi(x) \ = \ \varepsilon\varphi(x), \tag{2.17}$$

where the $P_j(x)$ are polynomials of jth order with coefficients related to $c_{\alpha\beta}, c_\alpha$ and n. In general, problem (2.17) has $(n + 1)$ polynomial solutions of the form of polynomials in x of nth degree. If $n = 1$, as a consequence of Theorem 2.2, a more general spectral problem than (2.17) arises (cf. (2.14))

$$-F_3(x)d_x^2\varphi(x) \ + \ q_2(x)d_x\varphi(x) \ + \ q_1(x)\varphi(x) \ = \ \varepsilon\varphi(x), \tag{2.18}$$

possessing only two polynomial solutions of the form $(ax + b)$. Here F_3 is an arbitrary complex function of x and $q_j(x), j = 1, 2$ are polynomials of order j the same as in (2.14). For the case $n = 0$ (one polynomial solution, $\varphi = const$) the spectral problem (2.5) becomes (cf. (2.13))

$$-F_2(x)d_x^2\varphi(x) \ + \ F_1(x)d_x\varphi(x) \ + \ q_0\varphi(x) \ = \ \varepsilon\varphi(x), \tag{2.19}$$

where $F_{2,1}(x)$ are arbitrary complex functions and $q_0 \in R$.

Substituting (2.1) into (2.15.2) and then into (2.6), we obtain

$$-Q_2(x)d_x^2\varphi(x) \ + \ Q_1(x)d_x\varphi(x) \ + \ Q_0(x)\varphi(x) \ = \ \varepsilon\varphi(x), \tag{2.20}$$

where the $Q_j(x)$ are polynomials of jth order with arbitrary coefficients. One can easily show that the differential operator in the r.h.s. can be always presented in the form (2.15.2). The coefficients of $Q_j(x)$ are unambiguously related to $c_{\alpha\beta}, c_\alpha$ for *any* value of the parameter n. Thus n is a fictitious parameter and, for instance, it can be put equal to zero.

2.3. **Quasi-exactly-solvable Schroedinger equations (examples).** From the point of applications the Schroedinger equation

$$(-d_z^2 \ + \ V(z))\Psi(z) \ = \ \varepsilon\Psi(z), \tag{2.21}$$

is one of the important among the second-order differential equations (see e.g. Landau and Lifschitz [1974]). Here ϵ is the spectral parameter and $\Psi(z)$ must be square-integrable function on some space. It can be the whole real line, semi-line or a finite interval. Therefore it is quite natural to search the quasi-exactly-solvable and exactly-solvable operators of the Schroedinger type acting on a finite-dimensional linear space of square-integrable functions:$f_i(z), i = 1, 2, \dots n$.

One possible way to get the quasi-exactly-solvable and exactly-solvable Schroedinger operators is to transform the quasi-exactly-solvable T_2 and exactly-solvable E_2 operators acting on finite-dimensional spaces of polynomials (1.5) into the Schroedinger type operators. It always can be done making a change of a variable and a gauge transformation (see (1.2)) as a consequence of one-dimensional nature of equations studied. In practice, the realization of this transformation is nothing but a conversion of (2.17)-(2.20) into (2.21). The only open question remains: does new basis belong to square-integrable one or not? This question will be discussed in the end of this Section. In the following consideration we restrict ourself by the case of real functions and real variables.

Introducing a new function

$$\Psi(z) \ = \ \varphi(x(z))e^{-A(z)}, \tag{2.22}$$

and new variable $x = x(z)$, where $A(z)$ is a certain real function, one can reduce (2.17)–(2.20) to the Sturm-Liouville-type problem (2.21) with the potential equal to

$$V(z) = (A')^2 - A'' + P_2(x(z)) \ , \tag{2.23}$$

if

$$A = \int (\frac{P_3}{P_4})dx - log z' \ , \ z = \pm \int \frac{dx}{\sqrt{P_4}} \ .$$

for the case of (2.17), or

$$A = \int (\frac{Q_2}{F_3})dx - log z' \ , \ z = \pm \int \frac{dx}{\sqrt{F_3}} \ .$$

for the case of (2.18), or

$$A = \int (\frac{F_1}{F_2})dx - log z' \ , \ z = \pm \int \frac{dx}{\sqrt{F_2}} \ .$$

for the case of (2.19) with replacement of $P_2(x(z))$ for two latter cases by $Q_1(x(z))$ or Q_0 , respectively. If the functions (2.22), obtained after transformation, belong to the $\mathcal{L}_2(\mathcal{D})$-space[2] , we arrive at the recently discovered quasi-exactly-solvable Schroedinger equations (Turbiner [1988a, 1988b, 1988c]), where a finite number of eigenstates is found algebraically.

In order to proceed a description of concrete examples of quasi-exactly-solvable Schroedinger equations, first of all let us generalize the eigenvalue problem (2.21) inserting a weight function $\hat{\varrho}(z) \equiv \varrho(x(z))$ in the r.h.s.

$$(-d_z^2 \ + \ V(z))\Psi(z) \ = \ \varepsilon\varrho(x(z))\Psi(z), \tag{2.24}$$

[2]Depending on the change of variable $x = x(z)$, the space $\mathcal{D}$ can be whole real line, semi-line and a finite interval.

where for a sake of future convenience the weight function is presented in the form of composition $\varrho(x(z))$. This equation can obtained from (2.17)-(2.19) by taking the same gauge factor (2.22) as before but with another change of the variable

$$z \;=\; \pm \int \frac{dx}{\sqrt{\varrho(x) P_4(x)}}$$

what leads to a slightly modified potential then in (2.23)

$$V(z) = (A')^2 - A'' + P_2(x(z))\varrho(x(z))$$

Below we will follow the catalogue given at Turbiner [1988c]. A presentation of results is the following: firstly, we display the quadratic element T_2 of the universal enveloping algebra sl_2 in the representation (2.1) and its equivalent form of differential operator $T_2(x, d_2)$, secondly, the corresponding potential $V(z)$and afterwards, the explicit expression for the change of the variable $x = x(z)$, the weight function $\varrho(z)$and finally the functional form of the eigenfunctions $\Psi(z)$ of the "algebraized" part of the spectra [3].

We begin a consideration with the quasi-exactly-solvable equations, associated to the exactly-solvable Morse oscillator; it implies that at the limit, when number of "algebraized" eigenstates $(n + 1)$ goes to infinity, the Morse oscillator occurs.

Comment 2.5 The Morse oscillator is one of well-known exactly-solvable quantum-mechanical problems (see e.g. Landau and Lifschitz [1974]). It is described by the Schroedinger operator with the potential

$$V(z) = A^2 e^{-2\alpha z} - 2Ae^{-\alpha z} \ , \ A, \alpha > 0 \ .$$

This potential is used to model the interaction of the atoms in diatomic molecules.

I.

$$T_2 = -\alpha^2 J_n^+ J_n^- + 2\alpha a J_n^+ - \alpha[\alpha(n+1) + 2b]J_n^0 - 2\alpha c J_n^- \\ -\frac{\alpha n}{2}[\alpha(n+1) + 2b] \tag{2.25}$$

or as the differential operator,

$$T_2(x, d_x) = -\alpha^2 x^2 d_x^2 + \alpha[2ax^2 - (\alpha + 2b)x - 2c]d_x - 2\alpha a n x$$

leads to

$$V(z) = a^2 e^{-2\alpha z} - a[2b + \alpha(2n+1)]e^{-\alpha z} + c(2b - \alpha)e^{\alpha z} + c^2 e^{2\alpha z} \\ +b^2 - 2ac \tag{2.26}$$

where

$$x = e^{-\alpha z} \ , \ \varrho = 1 \ ,$$

with the eigenfunctions of "algebraized" part of the spectra

$$\Psi(z) \;=\; p_n(e^{-\alpha z}) \exp{(-\frac{a}{\alpha} e^{-\alpha z} - bz - \frac{c}{\alpha} e^{\alpha z})}$$

at $\alpha > 0, \ a, c \geq 0$ and $\forall b$.

[3]The functions $p_n(x)$ occurring in the forthcoming expressions for $\Psi(z)$ are polynomials of the nth order. They are nothing but the polynomial eigenfunctions of the operator $T_2(x, d_x)$.

II.

$$T_2 = -\alpha J_n^0 J_n^- + 2aJ_n^+ - 2cJ_n^0 - [\alpha(\frac{n}{2}+1)+2b]J_n^- - cn \qquad (2.27)$$

or as the differential operator,

$$T_2(x,d_x) = -\alpha x d_x^2 + (-2ax^2 + 2cx + \alpha + 2b)d_x + 2anx$$

leads to

$$V(z) = a^2 e^{-4\alpha z} - 2ace^{-3\alpha z} + [c^2 - 2a(b+\alpha an+\alpha)]e^{-2\alpha z} + c(2b+\alpha)e^{-\alpha z} + b^2 \ , \qquad (2.28)$$

where

$$x = e^{-\alpha z} \ , \ \varrho = \frac{1}{\alpha}e^{-\alpha z} \ ,$$

$$\Psi(z) \ = \ p_n(e^{-\alpha z}) \exp(-\frac{a}{2\alpha}e^{-2\alpha z} + \frac{c}{\alpha}e^{-\alpha z} - bz)$$

at $\alpha > 0, a \geq 0, b > 0$ and $\forall c$.

III.

$$T_2 = -\alpha J_n^+ J_n^0 + (2b - 3\alpha n/2)J_n^+ - 2aJ_n^0 - 2cJ_n^- - an \qquad (2.29)$$

or as the differential operator,

$$T_2(x,d_x) = -\alpha x^3 d_x^2 + [(2b-\alpha)x^2 - 2ax - 2c]d_x + (\alpha n - 2b)nx$$

leads to

$$V(z) = c^2 e^{4\alpha z} + 2ace^{3\alpha z} + [a^2 - 2c(b+\alpha)]e^{2\alpha z} - a(2b+\alpha)e^{\alpha z} \ + b^2 + \alpha n(\alpha n - 2b) \ , \qquad (2.30)$$

where

$$x = e^{-\alpha z} \ , \ \varrho = \frac{1}{\alpha}e^{\alpha z} \ ,$$

$$\Psi(z) \ = \ p_n(e^{-\alpha z}) \exp(-\frac{c}{2\alpha}e^{2\alpha z} - \frac{a}{\alpha}e^{\alpha z} + bz)$$

at $\alpha > 0, b > 0, c \geq 0$ and $\forall a$.

Next two quasi-exactly-solvable potentials are associated to the one-soliton or Pöschle-Teller potential.

Comment 2.6 The Pöschle-Teller or one-soliton potential describes another well-known exactly-solvable quantum-mechanical problem (see e.g. Landau and Lifschitz [1974]). It is given by the Schroedinger operator with the potential

$$V(z) = -A^2 \frac{1}{\cosh^2 \alpha z} \ .$$

This potential has a unique property of an absence of reflection and is also one of the simplest solutions of so called Korteweg-de Vries equation playing important role in the *inverse problem method* (for detailed discussion see e.g. the book by V.E. Zakharov et al [1980]).

IV.

$$T_2 = -4\alpha^2 J_n^+ J_n^0 + 4\alpha^2 J_n^+ J_n^- - 2\alpha[\alpha(3n+2k+1)+2c]J_n^+$$
$$+ 2\alpha[\alpha(n+2)+2c-2a]J_n^0 + 4\alpha a J_n^- + \alpha n[\alpha(n+2)+2c-2a] \tag{2.31}$$

or as the differential operator,

$$T_2(x,d_x) = -4\alpha^2(x^3-x^2)d_x^2 - 2\alpha[(3\alpha+2\alpha k+2c)x^2 - 2(\alpha-a+c)x - 2a]d_x$$
$$+(2n+k)\alpha[(2n+k+1)\alpha+2c]x$$

leads to

$$V(z) = a^2\cosh^4\alpha z - a(a+2\alpha-2c)\cosh^2\alpha z \tag{2.32}$$
$$-[c(c+\alpha)+\alpha(2n+k)(\alpha(2n+k)+\alpha+2c)]\cosh^{-2}\alpha z + c^2 + a\alpha - 2ac$$

where

$$x = \cosh\alpha z^{-2}\ ,\ \varrho = 1\ ,$$
$$\Psi(z) \ = \ (\tanh\alpha z)^k p_n(\tanh^2\alpha z)(\cosh\alpha z)^{-c/\alpha}\exp\left(-\frac{a}{4\alpha}\cosh 2\alpha z\right)$$

at $\alpha > 0, a \geq 0, \forall c$ and $k = 0, 1$.

V.

$$T_2 = -4\alpha^2 J_n^+ J_n^- + 4\alpha^2 J_n^0 J_n^- + 4\alpha b J_n^+ - 2\alpha[\alpha(2n+2k+3)+2a+4b]J_n^0$$
$$+ 2\alpha[\alpha(n+2)+2a+2b)J_n^- - \alpha[\alpha n(2n+2k+3)+2an-2bk] \tag{2.33}$$

or as the differential operator,

$$T_2(x,d_x) = -4\alpha^2 x(x-1)d_x^2 + 2\alpha[2bx^2 - (2a+4b+2k\alpha+3\alpha)x + 2(\alpha+a+b)]d_x$$
$$-4\alpha bnx + 2\alpha b(2n+k)$$

leads to

$$V(z) = -b^2\cosh^{-6}\alpha z + b[2a+3b+\alpha(4n+2k+3)]\cosh^{-4}\alpha z \tag{2.34}$$
$$-[(a+3b)(a+b+\alpha)+2(2n+k)\alpha b)]\cosh^{-2}\alpha z + (a+b)^2$$

where

$$x = \cosh\alpha z^{-2}\ ,\ \varrho = \cosh^{-2}\alpha z\ ,$$
$$\Psi(z) \ = \ (\tanh\alpha z)^k p_n(\tanh^2\alpha z)(\cosh\alpha z)^{\frac{-(a+b)}{\alpha}}\exp\left(\frac{b}{2\alpha}\tanh^2 2\alpha z\right)$$

at $\alpha > 0, (a+b) > 0$ and $k = 0, 1$.

Next two quasi-exactly-solvable potentials are associated to the harmonic oscillator potential.

VI.

It is the first example of the quasi-exactly-solvable Schroedinger operator. Let us take the following non-linear combination in the generators (2.1) (Turbiner [1988c])

$$T_2 = -4J_n^0 J_n^- + 4aJ_n^+ + 4bJ_n^0 - 2(n+1+2k)J_n^- + 2bn \tag{2.35}$$

or as the differential operator,

$$T_2(x, d_x) = -4xd_x^2 + 2(2ax^2 + 2bx - 1 - 2k)d_x - 4anx \ ,$$

where $x \in R$ and $a > 0, \forall b$ or $a \geq 0, b > 0$. Putting $x = z^2$ and choosing the gauge phase $A = ax^2/4 + bx/2 - k/2 \ln x$, we arrive at the the spectral problem (2.19) with the potential (Turbiner and Ushveridze [1987])

$$V(z) = a^2 z^6 + 2abz^4 + [b^2 - (4n+3+2k)a]z^2 - b(1+2k), \tag{2.36}$$

for which at $k = 0$ ($k = 1$) the first $(n+1)$ eigenfunctions, even (odd) in x, can be found algebraically. Of course, the number of those "algebraized" eigenfunctions is nothing but the dimension of the irreducible representation (1.4) of the algebra (2.1). $(n+1)$ 'algebraized' eigenfunctions of (2.19) have the form

$$\Psi(z) \ = \ z^k p_n(z^2) e^{-\frac{az^4}{4} - \frac{bz^2}{2}}, \tag{2.37}$$

where $p_n(y)$ is a polynomial of the nth degree. It is worth noting that if the parameter a goes to 0, the potential (2.35) becomes the harmonic oscillator potential and polynomials $z^k p_n(z^2)$ reconcile to the Hermite polynomials $H_{2n+k}(z)$ (see discussion below).

VII.

$$T_2 = -4J_n^0 J_n^- + 4aJ_n^+ + 4bJ_n^0 - 2(n+d+2l-2c)J_n^- + 2bn \tag{2.38}$$

or as the differential operator,

$$T_2(x, d_x) = -4xd_x^2 + 2(2ax^2 + 2bx - d - 2l + 2c)d_x - 4anx$$

leads to

$$V(z) = a^2 z^6 + 2abz^4 + [b^2 - (4n+2l+d+2-2c)a]z^2 + c(c-2l-d+2)z^{-2} \tag{2.39}$$

$$-b(d+2l-2c),$$

where

$$x = z^2 \ , \ \varrho = 1 \ ,$$

$$\Psi(z) \ = \ p_n(z^2) z^{l-c} e^{-\frac{az^4}{4} - \frac{bz^2}{2}},$$

at $a > 0, \forall b$ or $a \geq 0, b > 0$ and $(d+l-c) > 1$. This case corresponds to the radial part of d-dimensional Schroedinger equation, $z \in [0, \infty)$. At $d = 1$ the radial part coincides with the ordinary Schroedinger equation (2.19) at $z \in R$. The potential (2.39) becomes a generalized version of the potential (2.36) with an additional singular term proportional to z^{-2}.

Next two quasi-exactly-solvable potentials are associated to the Coulomb problem.

VIII.

$$T_2 = -J_n^0 J_n^- + 2aJ_n^+ + 2bJ_n^0 - (n/2 + d + 2l - 2c - 1)J_n^- - an \tag{2.40}$$

or as the differential operator,

$$T_2(x, d_x) = -xd_x^2 + (2ax^2 + 2bx + 2c - d - 2l + 1)d_x - 2anx$$

leads to

$$V(z) = a^2z^2 + 2abz - b(2l + d - 1 - 2c)z^{-1} + c(c - 2l - d + 2)z^{-2} \tag{2.41}$$

$$+b^2 + a(2c - d - 2l - 2n) \ ,$$

where

$$x = z \ , \ \varrho = z^{-1} \ ,$$

$$\Psi(z) \ = \ p_n(z)z^{l-c}e^{-\frac{a}{2}z^2 - bz},$$

at $a \geq 0, b > 0$ and $(d + l - c) > 1$.

IX.

$$T_2 = -J_n^+ J_n^- + 2aJ_n^+ - (n + d - 1 + 2l - 2c)J_n^0 + 2bJ_n^- - n(d + 2l - 1 - 2c) \tag{2.42}$$

or as the differential operator,

$$T_2(x, d_x) = -x^2d_x^2 - [2ax^2 + (2c - d - 2l + 1)x - 2b]d_x - 4anx$$

leads to

$$V(z) = b^2z^{-4} - b(2c - 2l - d + 3)z^{-3} + [c(c - 2l - d + 2) - 2ab]z^{-2} \tag{2.43}$$

$$-a(2n + 2l + d - 1 - 2c)z^{-1} + a^2 \ ,$$

where

$$x = z \ , \ \varrho = z^{-2} \ ,$$

$$\Psi(z) \ = \ p_n(z)z^{l-c}e^{-az-bz^{-1}},$$

at $a \geq 0, b > 0$ or $a > 0, b \geq 0$ and $c, l, d \in R$.

Now let us show an example of the non-singular periodic quasi-exactly-solvable potential connected with Mathieu potential.

Comment 2.7 The Mathieu potential

$$V(z) \ = \ A\cos\alpha z$$

is one of the most important potentials in many branches of physics, engineering etc. Detailed description of the properties of the corresponding Scroedinger equation can be found, for instance, in Bateman-Erdélyi [1953], vol.3 and also Kamke [1959], Equation 2.22.

X.

Firstly, take the following operator

$$T_2 = \alpha^2 J^+_{n-\mu} J^-_{n-\mu} - \alpha^2 J^-_{n-\mu} J^-_{n-\mu} + \tag{2.44}$$

$$2a\alpha J^+_{n-\mu} + \alpha^2(n+\mu+1)J^0_{n-\mu} - 2\alpha a J^-_{n-\mu} + \alpha^2\frac{(n-\mu)(n+\mu+1)}{2}$$

or as the differential operator,

$$T_2(x,d_x) = \alpha^2(x^2-1)d_x^2 + \alpha[2ax^2 + \alpha(1+2\mu)x - 2a]d_x - 2\alpha a(n-\mu)x$$

The transformation (2.22)-(2.23) leads to the periodic potential

$$V(z) = a^2 \sin^2 \alpha z - \alpha a(2n+1)\cos\alpha z + \mu\alpha^2, \tag{2.45}$$

where

$$x = \cos\alpha z \ , \ \varrho = 1 \ ,$$
$$\Psi(z) \ = \ (\sin\alpha z)^\mu p_{n-\mu}(\cos\alpha z)\ e^{(\frac{a}{\alpha}\cos\alpha z)},$$

at $a \geq 0, \alpha \geq 0$. Here $\mu = 0,1$. For the fixed n, $(2n+1)$ eigenstates having a meaning of the edges of bands can be found algebraically.

Secondly, take the following operator

$$T_2 = \alpha^2 J^+_{n-1} J^-_{n-1} - \alpha^2 J^-_{n-1} J^-_{n-1} + \tag{2.46}$$

$$2a\alpha J^+_{n-1} + \alpha^2(n+1)J^0_{n-1} - \alpha[2a + \alpha(\nu_1-\nu_2)]J^-_{n-1} + \alpha^2\frac{(n^2-1)}{2}$$

or as the differential operator,

$$T_2(x,d_x) = \alpha^2(x^2-1)d_x^2 + \alpha[2ax^2 + 2\alpha x - 2a - \alpha(\nu_1-\nu_2)]d_x - 2\alpha a(n-1)x$$

The transformation (2.22)-(2.23) leads to the periodic potential

$$V(z) = a^2\sin^2\alpha z - \alpha a(2n)\cos\alpha z + \alpha a(\nu_1-\nu_2)] - \frac{\alpha^2}{4}\ , \tag{2.47}$$

where

$$x = \cos\alpha z \ , \ \varrho = 1 \ ,$$
$$\Psi(z) \ = \ (\cos\alpha z)^{\nu_1}(\sin\alpha z)^{\nu_2} p_{n-1}(\cos\alpha z)\ e^{(\frac{a}{\alpha}\cos\alpha z)},$$

at $a \geq 0, \alpha \geq 0$. Here $\nu_{1,2} = 0,1$, but $\nu_1+\nu_2 = 1$. For the fixed n, $(2n+1)$ eigenstates having a meaning of the edges of bands can be found algebraically.

2.4. **Quasi-exactly-solvable Schroedinger equations (Lame equation).** In this Section we consider one of the most important second-order ordinary differential equation – m-zone Lame equation

$$-\,d_x^2\Psi + m(m+1)\mathcal{P}(x)\Psi \ = \ \varepsilon\Psi \tag{2.48}$$

where $\mathcal{P}(x)$ is the Weierstrass function in a standard notation (see, e.g. Bateman and A. Erdélyi [1953]), which depends on two free parameters, and $m = 1,2,\dots$. In a description we mainly follow the paper (Turbiner [1989]).

The Weierstrass function is a double-periodic meromorphic function for which the equation $\mathcal{P}'^2 = (\mathcal{P}-e_1)(\mathcal{P}-e_2)(\mathcal{P}-e_3)$ holds, where $\sum e_i = 0$. Introducing the new variable $\xi = \mathcal{P}(x) + \frac{1}{3}\sum a_i$ in ((2.48)) (see, e.g. Kamke [1959]), the new equation emerges

$$\eta'' + \frac{1}{2}(\frac{1}{\xi-a_1} + \frac{1}{\xi-a_2} + \frac{1}{\xi-a_3})\eta' - \frac{m(m+1)\xi+\varepsilon}{4(\xi-a_1)(\xi-a_2)(\xi-a_3)}\eta = 0 \tag{2.49}$$

where $\eta(\xi) \equiv \psi(x)$. Here the new parameters a_i satisfy the system of linear equations $e_i = a_i - \frac{1}{3}\sum a_i$. Equation (2.49) is named by an algebraic form for the Lame equation. There exists a spectral parameter λ for which equation (2.49) has four types of solutions:

$$\eta^{(1)} = p_k(\xi) \tag{2.50.1}$$

$$\eta_i^{(2)} = (\xi - a_i)^{1/2} p_k(\xi) \quad , \quad i = 1,2,3 \tag{2.50.2}$$

$$\eta_i^{(3)} = (\xi - a_{l_1} l)^{1/2} (\xi - a_{l_2})^{1/2} p_{k-1}(\xi) \ , \quad l_1 \neq l_2; i \neq l_{1,2}; i, l_{1,2} = 1,2,3 \tag{2.50.3}$$

$$\eta^{(4)} = (\xi - a_1)^{1/2} (\xi - a_2)^{1/2} (\xi - a_3) p_{k-1}(\xi) \tag{2.50.4}$$

where $p_r(\xi)$ are polynomial in ξ of degree r. If the value of parameter n is fixed, there are $(2m+1)$ linear independent solutions of the following form: if $m = 2k$ is even, then the $\eta^{(1)}(\xi)$ and $\eta^{(3)}(\xi)$ solutions arise, if $m = 2k+1$ is odd we have solutions of the $\eta^{(2)}(\xi)$ and $\eta^{(4)}(\xi)$ types. Those eigenvalues have a meaning of the edges of the zones in the potential (2.48).

Theorem 2.3. *(Turbiner* [1989]*) The spectral problem (2.48) at $m = 1, 2, \ldots$ with polynomial solutions (2.50.1), (2.50.2), (2.50.3), (2.50.4) is equivalent to the spectral problem (2.5) for the operator T_2 (2.15.1) belonging the universal enveloping sl_2-algebra in the representation (2.1) with the coefficients*

$$c_{+0} = 4 \quad , \quad c_{+-} = -4\sum a_i \quad , \quad c_{0-} = 4\sum a_i a_j \quad , \quad c_{--} = a_1 a_2 a_3 \tag{2.51}$$

before the terms quadratic in generators and the following coefficients before linear in generators $J_r^{\pm,0}$:

(1) *For $\eta^{(1)}(\xi)$-type solutions at $m = 2k, r = k$*

$$c_+ = -6k - 2 \ , \ c_0 = 4(k+1)\sum a_i \ , \ c_- = -2(k+1)\sum a_i a_j \tag{2.52.1}$$

(2) *For $\eta_i^{(2)}(\xi)$-type solutions at $m = 2k+1, r = k$*

$$\begin{aligned} c_+ &= -6k - 6 \ , \ c_0 = 4(k+2)\sum a_i - a_i \ , \\ c_- &= -2(k+1)\sum a_i a_j - 4a_{l_1} a_{l_2} \ , \quad i \neq l_{1,2}, l_1 \neq l_2 \end{aligned} \tag{2.52.2}$$

(3) *For $\eta_i^{(3)}(\xi)$-type solutions at $m = 2k, r = k-1$*

$$\begin{aligned} c_+ &= -6k - 4 \ , \ c_0 = 4(k+1)\sum a_i + 4a_i \ , \\ c_- &= -2(k+2)\sum a_i a_j + 4a_{l_1} a_{l_2} \ , \quad i \neq l_{1,2}, l_1 \neq l_2 \end{aligned} \tag{2.52.3}$$

(4) *For $\eta_i^{(4)}(\xi)$-type solutions at $m = 2k+1, r = k-1$*

$$c_+ = -6k - 8 \ , \ c_0 = 4(k+2)\sum a_i \ , \ c_- = -2(k+2)\sum a_i a_j \tag{2.52.4}$$

So, each type of solution (2.50.1), (2.50.2), (2.50.3), (2.50.4) corresponds to the particular spectral problem (2.48) with a special set of parameters (2.51) plus (2.52.1), (2.52.2), (2.52.3), (2.52.4), respectively. It can be easily shown that the calculation of eigenvalues ε of (2.48) corresponds to the solution of a characteristic equation for the four-diagonal matrix:

$$\begin{aligned} C_{l,l-1} &= (l-1-2j)[4(j+1-l)+c_{+}] \ , \\ C_{l,l} &= [l(2j+1-l)c_{+-}+(l-j)c_{0}] \ , \\ C_{l,l+1} &= (l+1)(j-l)c_{0-}+(l+1)c_{-} \ , \\ C_{l,l+2} &= -(l+1)(l+2)c_{--}. \end{aligned} \tag{2.53}$$

where the size of this matrix is $(k+1)\times(k+1)$ and $2j=k$ for (2.50.1), (2.50.2), and $k\times k$ and $2j=k-1$ for (2.50.3), (2.50.4), respectively. In connection to Theorem 2.3 one prove the following theorem (Turbiner [1989]).

Theorem 2.4. *Let fix the parameters $e's$ $(a's)$ in (2.48) (or (2.49)) except one, e.g. $e_1(a_1)$. The first $(2m+1)$ eigenvalues of (2.48) (or (2.49)) form $(2m+1)$-sheeted Riemann surface in parameter $e_1(a_1)$. This surface contains four disconnected pieces: one of them corresponds to $\eta^{(1)}(\eta^{(4)})$ solutions and the others correspond to $\eta^{(3)}(\eta^{(2)})$. At $m=2k$ the Riemann subsurface for $\eta^{(1)}$ has $(k+1)$ sheets and the number of sheets in each of the others is equal to k. At $m=2k+1$ the number of sheets for $\eta^{(4)}$ is equal to k and for $\eta^{(2)}$ each subsurface contains $(k+1)$ sheets.*

It is worth emphasizing that we cannot find a relation between the spectral problem for the two-zone potential

$$V=-2\sum_{k=1}^{3}\mathcal{P}(x-x_i)\ , \quad \sum_{i=1}^{3}x_i=0\ , \tag{2.54}$$

(see Dubrovin and Novikov [1974]) [4] and the spectral problem (2.5) for T_2 with the parameters (2.51) and (2.52.1) or (2.52.3) at $k=1$. In this case eigenvalues ε and also eigenfunctions (2.49) (but not (2.48) do not depend on parameters c_{--}.

Comment 2.8 . One can generalize a meaning of isospectral deformation, saying we want to study a variety of potentials with the first several coinciding eigenvalues. It can be named *quasi-isospectral deformation.*

Now let us consider such a quasi-isospectral deformation of (2.48) at $m=2$. It arises from the fact that the addition of the term $c_{++}J_r^+J_r^+$ to the operator T_2 with the parameters (2.51) and (2.52.1) or (2.52.3) at $k=1$ does not change the characteristic matrix (2.53). Making the reduction (2.22)-(2.23) from the equation (2.5) to the Schroedinger equation (2.21), we obtain

$$V(x)=c_{++}\frac{2c_{++}\xi^6-c_{+-}\xi^4-2c_{0-}\xi^3-3c_{--}\xi^2}{P_4^2(\xi)}+P_2(\xi)\ , \tag{2.55}$$

where

$$P_4(\xi)=c_{++}\xi^4+c_{+0}\xi^3+c_{+-}\xi^2+c_{0-}\xi+c_{--}\ ,\ P_2(\xi)=-m(m+1)\xi+\frac{c_0}{2}\ . \tag{2.56}$$

[4] The potential (2.54) and the original Lame potential (2.48) at $m=2$ are related to via the isospectral deformation.

and ξ is defined via the equation

$$x = \int \frac{d\xi}{\sqrt{P_4(\xi)}} \ , \tag{2.57}$$

In general, the potential (2.55) contains four double poles in x and does not reduce to (2.54). It is worth noting that the first five eigenfunctions in the potential (2.55) have the form

$$\Psi(x) = \left\{ \begin{array}{c} A\xi + B \\ (\xi - a_i)^{1/2}(\xi - a_j)^{1/2} \end{array} \right\} \exp\left(-c_{++} \int \frac{\xi^3 d\xi}{P_4(\xi)} \right) \ , \ i \neq j, \ i, j = 1, 2, 3. \tag{2.58}$$

Here ξ is given by (2.57). The first five eigenvalues of the potential (2.55) do not depend on the parameters c_{--}, c_{++}.

2.5. **Exactly-solvable equations and classical orthogonal polynomials.** As one of the most important properties of the exactly-solvable operators (2.15.2) is the following: the eigenvalues of a general exactly-solvable operator E_2 are given by the quadratic polynomial in number of eigenstate

$$\epsilon_m = \ c_{00} m^2 \ + \ c_0 m \ + \ const \tag{2.59}$$

(for details see Turbiner [1988a, 1988b]). It can be easily verified by straightforward calculation.

Taking different exactly-solvable operators E_2 (see (2.15.2)) for the eigenvalue problem (2.5) one can reproduce the equations having the Hermite, Laguerre, Legendre and Jacobi polynomials as eigenfunctions (Turbiner [1988a, 1988b]), which is shown below. In the definition of the about polynomials we follow the definition given in (Bateman and A. Erdélyi [1953]).

1. Hermite polynomials.

The Hermite polynomials $H_{2m+k}, \ m = 0, 1, 2, \dots \ , \ k = 0, 1$ are the polynomial eigenfunctions of the operator

$$E_2(x, d_x) = d_x^2 - 2x d_x \tag{2.60}$$

which immediately can be rewritten in terms of the generators (2.1) following the Lemma 2.1

$$E_2 = \ J_0^-(x) J_0^-(x) \ - \ 2 J_0^0(x) \tag{2.61}$$

However, there exists another way to represent the operators related to the Hermite polynomials. Let us notice that k has a meaning of the parity of the polynomial H_{2m+k} and

$$H_{2m+k}(x) = x^k h_m(x^2)$$

Then it is easy to find that the operator having $h_m(y)$ as the eigenfunctions

$$\bar{E}_2(y, d_y) = 4y d_y^2 - 2(2y - 1 - 2k) d_y \tag{2.62}$$

and, correspondingly

$$\bar{E}_2 = \ 4J_0^0(y) J_0^-(y) \ - \ 4J_0^0(y) \ + \ 2(1 + 2k) J_0^-(y) \tag{2.63}$$

Of course, those two representations are equivalent, however, a quasi-exactly-solvable generalization can be implemented for the second representation only (see examples VI and VII in Section 2.3).

2. Laguerre polynomials.

The associated Laguerre polynomials $L_m^a(x)$ occur as the polynomial eigenfunctions of the generalized Laguerre operator

$$E_2(x,d_x) = xd_x^2 + (a+1-x)d_x \tag{2.64}$$

where a is any real number. Of course, the operator (2.64) can be rewritten as

$$E_2 = \ J_0^0 J_0^- \ - \ J_0^0 \ + \ (a+1)J_0^- \tag{2.65}$$

3. Legendre polynomials.

The Legendre polynomials $P_{2m+k}(x)$ are the the polynomial eigenfunctions of the operator

$$E_2(x,d_x) = (1-x^2)d_x^2 - 2xd_x \tag{2.66}$$

or,

$$E_2 = \ -J_0^0 J_0^0 \ + \ J_0^- J_0^- \ - \ J_0^0 \tag{2.67}$$

Analogously to the Hermite polynomials there exists another way to represent the operators related to the Legendre polynomials. Let us notice that k has a meaning of the parity of the polynomial P_{2m+k} and

$$P_{2m+k}(x) = x^k p_m(x^2)$$

Then it is easy to find that the operator having $p_m(y)$ as the eigenfunctions

$$\bar{E}_2(y,d_y) = 4y(1-y)d_y^2 + 2[1+2k-(3+2k)y]d_y \tag{2.68}$$

and, correspondently

$$\bar{E}_2 = \ -4J_0^+(y)J_0^-(y) \ + \ 4J_0^0(y)J_0^-(y) \ - \ 2(3+2k)J_0^0(y) \ + \ 2(1+2k)J_0^-(y) \tag{2.69}$$

4. Jacobi polynomials.

The Jacobi polynomials appear as the polynomial eigenfunctions of Jacobi equation taking in either symmetric form with the operator

$$E_2(x,d_x) = (1-x^2)d_x^2 + [b-a-(a+b+2)x]d_x \tag{2.70}$$

corresponding to

$$E_2 = \ -J_0^0 J_0^0 + J_0^- J_0^- - (1+a+b)J_0^0 + (b-a)J_0^-, \tag{2.71}$$

or asymmetric form (see e.g. the book by Murphy [1960] or Bateman–Erdélyi [1953])

$$E_2(x,d_x) = x(1-x)d_x^2 + [1+a-(a+b+2)x]d_x \tag{2.72}$$

corresponding to

$$E_2 = \ -J_0^0 J_0^0 + J_0^0 J_0^- - (1+a+b)J_0^0 + (a+1)J_0^-, \tag{2.73}$$

Under special choices of the general element $E_4(E_6, E_8)$, one can reproduce all known fourth-(sixth-, eighth-)order differential equations giving rise to infinite sequences of orthogonal polynomials (see e.g. Littlejohn [1988] and other papers in this volume).

Recently, A. González-Lopéz, N. Kamran and P. Olver [1993] gave the complete description of second-order polynomial elements of $U_{sl_2(\mathbf{R})}$ in the representation (2.1) leading to the square-integrable eigenfunctions of the Sturm-Liouville problem

(2.21) after transformation (2.22)-(2.23). Consequently, for second-order ordinary differential equation (2.17) the combination of their result and Theorems 2.1, 2.2 gives a general solution of the problem of classification of equations possessing a finite number of orthogonal polynomial solutions.

3. Finite-difference equations in one variable

3.1. **General consideration.** Let us define a multiplicative finite-difference operator, or a shift operator or the so-called Jackson symbol (see e.g. Exton [1983], Gasper and Rahman [1990])

$$Df(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{3.1}$$

where $q \in R$ and $f(x)$ is real function $x \in R$. The Leibnitz rule for the operator D is

$$Df(x)g(x) = (Df(x))g(x) + f(qx)Dg(x)$$

Now one can easily introduce the finite-difference analogue of the algebra of the differential operators (2.1) based on the operator D instead of the continuous derivative (Ogievetsky and Turbiner [1991])

$$\begin{aligned} \tilde{J}_n^+ &= x^2 D - \{n\}x \\ \tilde{J}_n^0 &= \ xD - \hat{n} \\ \tilde{J}_n^- &= \ D, \end{aligned} \tag{3.2}$$

where $\{n\} = \frac{1-q^n}{1-q}$ is so called q number and $\hat{n} \equiv \frac{\{n\}\{n+1\}}{\{2n+2\}}$. The operators (3.2) after multiplication by some factors

$$\tilde{j}^0 = \frac{q^{-n}}{p+1}\frac{\{2n+2\}}{\{n+1\}}\tilde{J}_n^0$$

$$\tilde{j}^{\pm} = q^{-n/2}\tilde{J}_n^{\pm}$$

(see Ogievetsky and Turbiner [1991]) form a quantum algebra $sl_2(\mathbf{R})_q$ with the following commutation relations

$$\begin{aligned} q\tilde{j}^0\tilde{j}^- \ - \ \tilde{j}^-\tilde{j}^0 \ &= \ -\tilde{j}^- \\ q^2\tilde{j}^+\tilde{j}^- \ - \ \tilde{j}^-\tilde{j}^+ \ &= \ -(q+1)\tilde{j}^0 \\ \tilde{j}^0\tilde{j}^+ \ - \ q\tilde{j}^+\tilde{j}^0 \ &= \ \tilde{j}^+ \end{aligned} \tag{3.3}$$

The parameter q does characterize the deformation of the commutators of the classical Lie algebra sl_2. If $q \to 1$, the commutation relations (3.3) reduce to the standard $sl_2(\mathbf{R})$ ones. A remarkable property of generators (3.2) is that, *if n is a non-negative integer, they form the finite-dimensional representation corresponding the finite-dimensional representation space $\mathcal{P}_{n+1}$ the same as of the non-deformed* sl_2 (see (2.1)). For values of q others than root of unity this representation is irreducible.

Comment 3.1 The algebra (3.3) is known in literature as so-called *the second Witten quantum deformation* of sl_2 in the classification of C. Zachos [1991]).

Similarly as for differential operators one can introduce quasi-exactly-solvable $\tilde{T}_k(x, D)$ and exactly-solvable finite-difference operators $\tilde{E}_k(x, D)$ (see definition 2.1).

Lemma 3.1. *(Turbiner* [1994b]*) (i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator $\tilde{T}_k$, can be represented by a kth degree polynomial of the operators (3.2). If $n \le (k-1)$, the part of the quasi-exactly-solvable operator $\tilde{T}_k$ containing derivatives up to order n can be represented by a nth degree polynomial in the generators (3.2).*

(ii) Conversely, any polynomial in (3.2) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $\tilde{E}_k \subset \tilde{T}_k$.

Comment 3.2 If we define an analogue of the universal enveloping algebra U_g for the quantum algebra $\tilde{g}$ as an algebra of all ordered polynomials in generators, then a quasi-exactly-solvable operator $\tilde{T}_k$ at $k < n+1$ is simply an element of the 'universal enveloping algebra' $U_{sl_2(\mathbf{R})_q}$ of the algebra $sl_2(\mathbf{R})_q$ taken in representation (3.2). If $k \geq n+1$, then $\tilde{T}_k$ is represented as an element of $U_{sl_2(\mathbf{R})_q}$ plus BD^{n+1}, where B is any linear difference operator of order not higher than $(k-n-1)$.

Similar to $sl_2(\mathbf{R})$ (see definition 2.3), one can introduce the grading of generators (3.2) of $sl_2(\mathbf{R})_q$ (cf. (2.3)) and, hence, the grading of monomials of the universal enveloping $U_{sl_2(\mathbf{R})_q}$ (cf. (2.4)).

Lemma 3.2. *A quasi-exactly-solvable operator* $\tilde{T}_k \subset U_{sl_2(\mathbf{R})_q}$ *has no terms of positive grading, iff it is an exactly-solvable operator.*

Theorem 3.1. *(Turbiner* [1994b]*) Let n be a non-negative integer. Take the eigenvalue problem for a linear difference operator of the k-th order in one variable*

$$\tilde{T}_k(x, D)\varphi(x) \ = \ \varepsilon\varphi(x), \tag{3.4}$$

where $\tilde{T}_k$ *is symmetric. The problem (3.4) has* $(n+1)$ *linearly independent eigenfunctions in the form of a polynomial in variable x of order not higher than n, if and only if* T_k *is quasi-exactly-solvable. The problem (3.4) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable* $\tilde{E}_k$.

Comment 3.2 Saying the operator $\tilde{T}_k$ is symmetric, we imply that, considering the action of this operator on a space of polynomials of degree not higher than n, one can introduce a positively-defined scalar product, and the operator $\tilde{T}_k$ is symmetric with respect to it.

This theorem gives a general classification of finite-difference equations

$$\sum_{j=0}^{k} \tilde{a}_j(x) D^j \varphi(x) \ = \ \varepsilon\varphi(x) \tag{3.5}$$

having polynomial solutions in x. The coefficient functions must have the form

$$\tilde{a}_j(x) \ = \ \sum_{i=0}^{k+j} \tilde{a}_{j,i} x^i . \tag{3.6}$$

In particular, this form occurs after substitution (3.2) into a general kth degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})_q}$. It guarantees the existence of at least a finite number of polynomial solutions. The coefficients $\tilde{a}_{j,i}$ are related to the coefficients of the kth degree polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})_q}$. The number of free parameters of the polynomial solutions is defined by the number of free parameters of a general k-th order polynomial element of the universal enveloping algebra $U_{sl_2(\mathbf{R})_q}$.[5] A rather straightforward

[5]For quantum $sl_2(\mathbf{R})_q$ algebra there are no polynomial Casimir operators (see, e.g. Zachos [1991]). However, in the representation (3.2) the relationship between generators analogous to the quadratic Casimir operator

$$q\tilde{J}_n^+ \tilde{J}_n^- - \tilde{J}_n^0 \tilde{J}_n^0 + (\{n+1\} - 2\hat{n})\tilde{J}_n^0 = \hat{n}(\hat{n} - \{n+1\})$$

calculation leads to the following formula

$$par(\tilde{T}_k) = (k+1)^2 + 1$$

(for the second-order finite-difference equation $par(\tilde{T}^2) = 10$). For the case of an infinite sequence of polynomial solutions the formula (3.6) simplifies to

$$\tilde{a}_j(x) \ = \ \sum_{i=0}^{j} \tilde{a}_{j,i} x^i \tag{3.7}$$

and the number of free parameters is given by

$$par(\tilde{E}_k) = \frac{(k+1)(k+2)}{2} + 1$$

(for $k = 2$, $par(\tilde{E}^2) = 7$). The increase in the number of free parameters compared to ordinary differential equations is due to the presence of the deformation parameter q.

3.2. **Second-order finite-difference exactly-solvable equations.** In Turbiner [1992] it is implemented a description in the present approach of the q-deformed Hermite, Laguerre, Legendre and Jacobi polynomials (for definitions of these polynomials see Exton [1983], Gasper and Rahman [1990]). In order to reproduce the known q-deformed classical Hermite, Laguerre, Legendre and Jacobi polynomials (for the latter, there exists the q-deformation of the asymmetric form (2.61) only, see e.g. Exton [1983], Gasper and Rahman [1990]), one should modify the spectral problem (3.4):

$$\tilde{T}_k(x, D)\varphi(x) \ = \ \varepsilon\varphi(qx), \tag{3.8}$$

by introducing the r.h.s. function the dependence on the argument qx (cf. (2.5) and (3.4)) as it follows from the book by Exton [1983] (see also Gasper and Rahman [1990]). Then corresponding q-difference operators having q-deformed classical Hermite, Laguerre, Legendre and Jacobi polynomials as eigenfunctions (see the equations (5.6.2), (5.5.7.1), (5.7.2.1), (5.8.3) in Exton [1983], respectively) are given by the combinations in the generators:

$$\tilde{E}_2 = \tilde{J}_0^- \tilde{J}_0^- \ - \ \{2\}\tilde{J}_0^0, \tag{3.9.1}$$

$$\tilde{E}_2 = \tilde{J}_0^0 \tilde{J}_0^- \ - \ q^{-a-1}\tilde{J}_0^0 \ + \ (q^{-a-1}\{a+1\})\tilde{J}_0^-, \tag{3.9.2}$$

$$\tilde{E}_2 = -q\tilde{J}_0^0\tilde{J}_0^0 + \tilde{J}^-\tilde{J}^- + (q - \{2\})\tilde{J}_0^0, \tag{3.9.3}$$

$$\tilde{E}_2 = -q^{a+b-1}\tilde{J}_0^0\tilde{J}_0^0 + q^a\tilde{J}_0^0\tilde{J}_0^- + [q^{a+b-1} - \{a\}q^b - \{b\}]\tilde{J}_0^0 + \{a\}\tilde{J}_0^- \ , \tag{3.9.4}$$

respectively.

Lemma 3.3. *If the operator $\tilde{T}_2$ (for the definition, see (2.15.1)) is such that*

$$\tilde{c}_{++} = 0 \quad and \quad \tilde{c}_+ = (\hat{n} - \{m\})\tilde{c}_{+0} \ , \ at\ some\ m = 0, 1, 2, \ldots \tag{3.10}$$

then the operator $\tilde{T}_2$ preserves both $\mathcal{P}_{n+1}$ and $\mathcal{P}_{m+1}$, and polynomial solutions in x with 8 free parameters occur.

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appears. It reduces the number of independent parameters of the second-order polynomial element of $U_{sl_2(\mathbf{R})_q}$. It becomes the standard Casimir operator at $q \to 1$.

As usual in quantum algebras, a rather outstanding situation occurs if the deformation parameter q is equal to a primitive root of unity. For instance, the following statement holds.

Lemma 3.4. *If a quasi-exactly-solvable operator $\tilde{T}_k$ preserves the space $\mathcal{P}_{n+1}$ and the parameter q satisfies to the equation*

$$q^n \;=\; 1 \;, \tag{3.11}$$

then the operator $\tilde{T}_k$ preserves an infinite flag of polynomial spaces $\mathcal{P}_0 \subset \mathcal{P}_{n+1} \subset \mathcal{P}_{2(n+1)} \subset \cdots \subset \mathcal{P}_{k(n+1)} \subset \ldots$.

It is worth emphasizing that, in the limit as q tends to one, Lemmas 3.1,3.2,3.3 and Theorem 3.1 coincide with Lemmas 2.1,2.2,2.3 and Theorem 2.1, respectively. Thus the case of differential equations in one variable can be treated as a limiting case of finite-difference ones. Evidently, one can consider the finite-difference operators, which are a mixture of generators (3.2) with the same value of n and different q's.

4. 2×2 MATRIX DIFFERENTIAL EQUATIONS ON THE REAL LINE

4.1. **General consideration.** This Section is devoted to a description of quasi-exactly-solvable , $\mathbf{T}_k(x)$ and exactly-solvable, $\mathbf{E}_k(x)$, 2×2 matrix differential operators acting on space of the two-component spinors with polynomial components

$$\mathcal{P}_{n+1,m+1} \ = \ \left\langle \begin{array}{c} x^0, x^1, \ldots, x^m \\ x^0, x^1, \ldots, x^n \end{array} \right\rangle \tag{4.1}$$

This space is a natural generalization of the space $\mathcal{P}_n$ (see (1.5)). The definition of quasi- and exactly-solvable operators is also a natural generalization of the Definition 2.1 .

Now let us introduce following two sets of 2×2 matrix differential operators:

$$\begin{aligned} T^+ &= x^2\partial_x - nx + x\sigma^+\sigma^-, \\ T^0 &= x\partial_x - \frac{n}{2} + \frac{1}{2}\sigma^+\sigma^- \ , \\ T^- &= \partial_x \ . \\ J &= -\frac{n}{2} - \frac{1}{2}\sigma^+\sigma^- \end{aligned} \tag{4.2}$$

named bosonic (even) generators and

$$Q \ = \ \begin{bmatrix} \sigma^- \\ x\sigma^- \end{bmatrix} \ , \ \bar{Q} \ = \ \begin{bmatrix} x\sigma^+\partial_x - n\sigma^+ \\ -\sigma^-\partial_x \end{bmatrix}. \tag{4.3}$$

named fermionic (odd) generators, where $\sigma^{\pm,0}$ are Pauli matrices in standard notation

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

It is easy to check that these generators form the algebra $osp(2,2)$:

$$\begin{gathered} [T^0, T^{\pm}] = \pm T^{\pm} \quad , \quad [T^+, T^-] = -2T^0 \quad , \quad [J, T^{\alpha}] = 0 \quad , \alpha = +, -, 0 \\ \{Q_1, \overline{Q}_2\} = -T^- \quad , \quad \{Q_2, \overline{Q}_1\} = T^+ \quad , \\ \frac{1}{2}(\{\overline{Q}_1, Q_1\} + \{\overline{Q}_2, Q_2\}) = +J \quad , \quad \frac{1}{2}(\{\overline{Q}_1, Q_1\} - \{\overline{Q}_2, Q_2\}) = T^0 \quad , \\ \{Q_1, Q_1\} = \{Q_2, Q_2\} = \{Q_1, Q_2\} = 0 \quad , \\ \{\overline{Q}_1, \overline{Q}_1\} = \{\overline{Q}_2, \overline{Q}_2\} = \{\overline{Q}_1, \overline{Q}_2\} = 0 \quad , \\ [Q_1, T^+] = Q_2 \quad , \quad [Q_2, T^+] = 0 \quad , \quad [Q_1, T^-] = 0 \quad , \quad [Q_2, T^-] = -Q_1 \ , \\ [\overline{Q}_1, T^+] = 0 \quad , \quad [\overline{Q}_2, T^+] = -\overline{Q}_1 \quad , \quad [\overline{Q}_1, T^-] = \overline{Q}_2 \quad , \quad [\overline{Q}_2, T^-] = 0 \ , \\ [Q_{1,2}, T^0] = \pm\frac{1}{2}Q_{1,2} \quad , \quad [\overline{Q}_{1,2}, T^0] = \mp\frac{1}{2}\overline{Q}_{1,2} \\ [Q_{1,2}, J] = -\frac{1}{2}Q_{1,2} \quad , \quad [\overline{Q}_{1,2}, J] = \frac{1}{2}\overline{Q}_{1,2} \end{gathered} \tag{4.4}$$

This algebra contains the algebra $sl_2(\mathbf{R}) \oplus \mathbf{R}$ as sub-algebra.

Lemma 4.1. *(Turbiner* [1994b]*) Consider the space $\mathcal{P}_{n+1,n}$.*

(i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator $\mathbf{T}_k(x)$, can be represented by a kth degree polynomial of the operators (4.2), (4.3). If $n \le (k-1)$, the part of the quasi-exactly-solvable operator $\mathbf{T}_k(x)$ containing derivatives in x up to order n can be represented by an nth degree polynomial in the generators (4.2), (4.3).

(ii) Conversely, any polynomial in (4.2), (4.3) is a quasi-exactly solvable operator.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $\mathbf{E}_k \subset \mathbf{T}_k(x)$.

Let us introduce the grading of the bosonic generators (4.2)

$$deg(T^+) = +1\ ,\ deg(J, T^0) = 0\ ,\ deg(J^-) = -1 \tag{4.5}$$

and fermionic generators (4.3)

$$deg(Q_2, \overline{Q}_1) = +\frac{1}{2}\ ,\ deg(Q_1, \overline{Q}_2) = -\frac{1}{2} \tag{4.6}$$

Hence the grading of monomials of the generators (31), (32) is equal to

$$\begin{aligned} deg[(T^+)^{n_+}(T^0)^{n_0}(J)^{\overline{n}}(T^-)^{n_-}{Q_1}^{m_1}{Q_2}^{m_2}{\overline{Q}_1}^{\overline{m}_1}{\overline{Q}_2}^{\overline{m}_2}] \ &= \\ (n_+ - n_-)\ -\ (m_1 - m_2 - \overline{m}_1 + \overline{m}_2)/2 & \end{aligned} \tag{4.7}$$

The n's can be arbitrary non-negative integers, while the m's are equal to either 0 or 1. The notion of grading allows us to classify the operators $\mathbf{T}_k(x)$ in the Lie-algebraic sense.

Lemma 4.2. *A quasi-exactly-solvable operator $\mathbf{T}_k(x) \subset U_{osp(2,2)}$ has no terms of positive grading other than monomials of grading +1/2 containing the generator Q_1 or Q_2, iff it is an exactly-solvable operator.*

Take the eigenvalue problem

$$\mathbf{T}_k(x)\varphi(x)\ =\ \varepsilon\varphi(x) \tag{4.8}$$

where

$$\varphi(x)\ =\ \begin{bmatrix}\varphi_1(x)\\ \varphi_2(x)\end{bmatrix}, \tag{4.9}$$

is a two-component spinor.

Theorem 4.1. *(Turbiner* [1994b]*) Let n be a non-negative integer. Take the eigenvalue problem (4.7), where $\mathbf{T}_k(x)$ is symmetric. In general, the problem (4.7) has $(2n+1)$ linearly independent eigenfunctions in the form of polynomials $\mathcal{P}_{n+1,n}$, if and only if $\mathbf{T}_k(x)$ is quasi-exactly-solvable. The problem (4.7) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable.*

As a consequence of Theorem 4.1, the 2×2 matrix quasi-exactly-solvable differential operator $\mathbf{T}_k(x)$, possessing in general $(2n+1)$ polynomial eigenfunctions of the form $\mathcal{P}_{n,n-1}$ can be written in the form

$$\mathbf{T}_k(x)\ =\ \sum_{i=0}^{i=k} \mathbf{a}_{k,i}(x) d_x^i\ , \tag{4.10}$$

The coefficient functions $\mathbf{a}_{k,i}(x)$ are by 2×2 matrices and generically for the kth order quasi-exactly-solvable operator their matrix elements are polynomials. Suppose that $k>0$. Then the matrix elements are given by the following expressions

$$\mathbf{a}_{k,i}(x) \ = \ \begin{pmatrix} A_{k,i}^{[k+i]} & B_{k,i}^{[k+i-1]} \\ C_{k,i}^{[k+i+1]} & D_{k,i}^{[k+i]} \end{pmatrix} \tag{4.11}$$

at $k>0$, where the superscript in square brackets displays the order of the corresponding polynomial.

It is easy to calculate the number of free parameters of a quasi-exactly-solvable operator $\mathbf{T}_k(x)$

$$par(\mathbf{T}_k(x)) = 4(k+1)^2 \tag{4.12}$$

For the case of exactly-solvable problems, the matrix elements (4.10) of the coefficient functions are modified

$$\mathbf{a}_{k,i}(x) \ = \ \begin{pmatrix} A_{k,i}^{[i]} & B_{k,i}^{[i-1]} \\ C_{k,i}^{[i+1]} & D_{k,i}^{[i]} \end{pmatrix} \tag{4.13}$$

where $k>0$. An infinite family of orthogonal polynomials as eigenfunctions of Eqs. (4.8)-(4.9) , if they exist, will occur, if and only if the coefficient functions have the form (4.12). The number of free parameters of an exactly-solvable operator $\mathbf{E}_k(x)$ and, correspondingly, the maximal number of free parameters of the 2×2 matrix orthogonal polynomials in one real variable, is equal to

$$par(\mathbf{E}_k(x)) = 2k(k+3)+3\ . \tag{4.14}$$

Thus, the above formulas describe the coefficient functions of matrix differential equation (4.8), which can possess polynomials in x as solutions.

4.2. **Quasi-exactly-solvable matrix Schroedinger equations (example).** Now let us take the quasi-exactly-solvable matrix operator $\mathbf{T}_2(x)$ and try to reduce Eq. (4.8) to the Schroedinger equation

$$[-\frac{1}{2}\frac{d^2}{dy^2}+\mathbf{V}(y)]\Psi(y) \ = \ E\Psi(y) \tag{4.15}$$

where $\mathbf{V}(y)$ is a two-by-two *hermitian* matrix, by making a change of variable $x\mapsto y$ and "gauge" transformation

$$\Psi(y) \ = \ \mathbf{U}(y)\varphi(x(y)) \tag{4.16}$$

where $\mathbf{U}$ is an arbitrary 2×2 matrix depending on the variable y. In order to get some "reasonable" Schroedinger equation one should fulfill two requirements: (i) the potential $\mathbf{V}(y)$ must be hermitian and (ii) the eigenfunctions $\Psi(y)$ must belong to a certain Hilbert space.

Unlike the case of quasi-exactly-solvable differential operators in one real variable (see González-Lopéz, Kamran and Olver[1993]), this problem has no complete solution so far. Therefore it seems instructive to display a particular non-trivial example of the quasi-exactly-solvable 2×2 -matrix Schroedinger operator (Shifman and Turbiner [1989]).

Take the quasi-exactly-solvable operator

$$\mathbf{T}_2 = -2T^0T^- + 2T^-J - i\beta T^0 Q_1 +$$

$$\alpha T^0 - (2n+1)T^- - \frac{i\beta}{2}(3n+1)Q_1 + \frac{i}{2}\alpha\beta Q_2 - i\beta\overline{Q}_1\ , \tag{4.17}$$

where α and β are parameters. Upon introducing a new variable $y = x^2$ and after straightforward calculations one finds the following expression for the matrix U in Eq. (4.16)

$$\mathbf{U} = \exp(-\frac{\alpha y^2}{4} + \frac{i\beta y^2}{4}\sigma_1) \tag{4.18}$$

and for the potential $\mathbf{V}$ in Eq. (4.15)

$$\mathbf{V}(y) = \frac{1}{8}(\alpha^2 - \beta^2)y^2 + \sigma_2[-(n + \frac{1}{4})\beta + \frac{\alpha\beta}{4}y^2 - \frac{\alpha}{4}\tan\frac{\beta y^2}{2}]\cos\frac{\beta y^2}{2} +$$
$$\sigma_3[-(n + \frac{1}{4})\beta + \frac{\alpha\beta}{4}y^2 - \frac{\alpha}{4}\cot\frac{\beta y^2}{2}]\sin\frac{\beta y^2}{2} \tag{4.19}$$

It is easy to see that the potential $\mathbf{V}$ is hermitian; $(2n+1)$ eigenfunctions have the form of polynomials multiplied by the exponential factor U and they are obviously normalizable.

Recent development of matrix quasi-exactly-solvable differential equations and further explicit examples can be found in Brihaye-Kosinski [1994].

5. Ordinary differential equations with the parity operator

Let us introduce the parity operator K in the following way

$$Kf(x) = f(-x) \tag{5.1}$$

The operator K has such properties

$$\{K, x\}_+ \ = 0 \ , \ \{K, d_x\}_+ \ = 0 \ , \ K^2 \ = \ 1. \tag{5.2}$$

Now take the following generators

$$J^-(n) = d_x + \frac{\nu}{x}(1-K) \ , \tag{5.3}$$

$$J_1^0(n) = xJ^-(n) = xd_x + \nu(1-K) \ , \tag{5.4}$$

$$J^0(n) = n - xd_x \ , \tag{5.5}$$

$$J^+(n) = xJ^0(n) = nx - x^2 d_x \ . \tag{5.6}$$

These operators together with the operator K form an algebra, which was named $gl(2,\mathbf{R})_K$ in Turbiner [1994a], with commutation relations

$$[J^0(n), J_1^0(n)] = 0 \ , \tag{5.7}$$

$$[J^\pm(n), J^0(n)] = \pm J_i^\pm(n) \ , \tag{5.8}$$

$$[J^\pm(n), J_1^0(n)] = \mp(1 \mp 2\nu K)J^\pm(n) \ , \tag{5.9}$$

$$[J^+(n), J^-(n)] = J_1^0(n) - (1+2\nu K)J^0(n) \tag{5.10}$$

and also

$$[K, J_1^0(n)] \ = \ [K, J^0(n)] \ = \ 0 \ , \{K, J^\pm(n)\}_+ \ = \ 0 \ . \tag{5.11}$$

If $\nu = 0$ the generators (5.3)-(5.6) and commutation relations (5.7)-(5.10) become those of the algebra $gl(2,\mathbf{R})$ [6].

It is easy to check, that the algebra (5.7)–(5.11) possesses two Casimir operators – the operators commuting with all generators of the algebra $gl(2,\mathbf{R})_K$: the linear one

$$C_1 = J^0 + J_1^0 + \nu K \tag{5.12}$$

and the quadratic one

$$C_2 = \frac{1}{2}\{J^+, J^-\}_+ + \frac{1}{4}(J^0 - J_1^0)^2 - \frac{\nu}{2}(J^0 - J_1^0)K + \frac{\nu}{2}K \ . \tag{5.13}$$

Substituting the concrete representation (5.3)–(5.6) into (5.12)–(5.13) , one can find that

$$C_1 = n + \nu \ ,$$

$$C_2 = \frac{1}{4}[(n+\nu)(n+\nu+2) - \nu^2] \tag{5.14}$$

The representation (5.3)-(5.6) has an outstanding property: for generic ν, when n is a non-negative integer number, there appears a finite-dimensional irreducible

[6] Hereafter we will omit the argument n in the generators except for the cases where the representation (5.3)–(5.6) is used concretely

representation $\mathcal{P}_{n+1}$ of dimension $(n+1)$. So the representation space remains the same for any value of parameter ν.

Like it has been done before one can introduce quasi-exactly- and exactly-solvable mixed operators, containing the differential operator and the parity operator K : $T_k(x, d_x, K)$ and $E_k(x, d_x, K)$, respectively. It is evident, that those operators are linear in K.

Lemma 5.1. *(i) Suppose $n > (k-1)$. Any quasi-exactly-solvable operator T_k, can be represented by a k-th degree polynomial of the operators (5.3) - (5.6) If $n \leq (k-1)$, the part of the quasi-exactly-solvable operator T_k containing derivatives up to order n can be represented by an nth degree polynomial in the generators (5.3)-(5.6).*

(ii) Conversely, any polynomial in (5.3)-(5.6) is quasi-exactly solvable.

(iii) Among quasi-exactly-solvable operators there exist exactly-solvable operators $E_k \subset T_k$.

Similarly to the case of $sl_2(\mathbf{R})$ one can introduce the grading of generators (5.3)-(5.6)

$$deg(J^+(n)) = +1 \ , \ deg(J^0(n), J^0_1(n), K) = 0 \ , \ deg(J^-(n)) = -1, \tag{5.15}$$

and

$$deg[(J^+(n))^{k_+}(J^0(n))^{k_0}(J^0_1(n))^{k_{0,1}}(K)^k(J^-(n))^{k_-}] \ = \ k_+ - k_- . \tag{5.16}$$

(cf. (2.3)–(2.4)). The grading allows us to classify the operators T_k in the algebraic sense.

Lemma 5.2. *A quasi-exactly-solvable operator $T_k \subset U_{gl_2(\mathbf{R})_K}$ has no terms of positive grading, if and only if it is an exactly-solvable operator.*

Theorem 5.1. *Let n be a non-negative integer. Take the eigenvalue problem for a linear differential operator of the kth order in one variable*

$$T_k(x, d_x, K)\varphi \ = \ \varepsilon\varphi \ , \tag{5.17}$$

where T_k is symmetric. The problem (5.17) has $(n+1)$ linearly independent eigenfunctions in the form of a polynomial in variable x of order not higher than n, if and only if T_k is quasi-exactly-solvable. The problem (5.17) has an infinite sequence of polynomial eigenfunctions, if and only if the operator is exactly-solvable. If T_k has no terms of odd grading, $(k_+ - k_-)$ is odd number (see (5.16)), T_k commutes with K and eigenfunctions in (5.17) have definite parity with respect to $x \to -x$.

Following the Lemma 5.1 a general second-order quasi-exactly-solvable differential operator is defined by a quadratic polynomial in generators of $gl_2(\mathbf{R})_K$. Provided that the conditions (5.12)–(5.14) are taken into account [7] we arrive at

[7] It leads to a disappearance of the terms containing, for instance, $J^0_1(n)$ and $J^0(n)J^0(n)$

$$T_2 = c_{++}J^+(n)J^+(n) + c_{+0}J^+(n)J^0(n) + c_{+-}J^+(n)J^-(n)+$$

$$c_{0-}J^0(n)J^-(n) + c_{--}J^-(n)J^-(n) + c_+J^+(n) + c_0J^0(n) + c_-J^-(n) + c, \tag{5.18.1}$$

(cf. (2.15.1), where $c_{\alpha\beta} = C^0_{\alpha\beta} + C^K_{\alpha\beta}, c_\alpha = C^0_\alpha + C^K_\alpha, c = C^0 + C^K$ and all C's are real numbers. The number of free parameters is $par(T_2) = 18$. Non-existence in T_2 of the terms of odd grading leads to the conditions

$$c_{+0} \ = \ c_{0-} \ = \ 0$$

and

$$c_+ \ = \ c_- \ = \ 0$$

and, finally, a general operator having eigenfunctions of a definite parity is

$$T_2^{(e,o)} = c_{++}J^+(n)J^+(n) + c_{+-}J^+(n)J^-(n) + c_{--}J^-(n)J^-(n) + c_0J^0(n) + c, \tag{5.18.2}$$

and the number of free parameters is $par(T_2^{(e.o)}) = 10$.

The condition of Lemma 5.2 requires that

$$c_{++} \ = \ c_{+0} \ = \ c_+ \ = \ 0 \ ,$$

and then the operator T_2 becomes exactly-solvable

$$E_2 = c_{+-}J^+(n)J^-(n) + c_{0-}J^0(n)J^-(n) + c_{--}J^-(n)J^-(n) + c_0J^0(n) + c_-J^-(n) + c, \tag{5.18.3}$$

(cf. (2.15.2) and the number of free parameters is reduced to $par(E_2) = 12$.

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Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail address: `TURBINER@CERNVM` or `TURBINER@VXCERN.CERN.CH`
Current address: Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, 01000 México D.F., MEXICO